Contact Equivalence of the Generalized Hunter - Saxton Equation and the Euler - Poisson Equation

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Abstract. We present a contact transformation of the generalized Hunter–Saxton equation to the Euler–Poisson equation with special values of the Ovsiannikov invariants. We also find the general solution for the generalized Hunter–Saxton equation.

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The generalized Hunter–Saxton equation

$$u_{tx} = u\, u_{xx} + \kappa\, u_x^2 \tag{1}$$

has a number of applications in the nonlinear instability theory of a director field of a liquid crystal, [1], in geometry of Einstein–Weil spaces, [2, 3], in constructing partially invariant solutions for the Euler equations of an ideal fluid, [4], and has been a subject of many recent studies. In the case $\kappa = \frac{1}{2}$ the general solution, [1], the tri-Hamiltonian formulation, [5], the pseudo-spherical formulation and the quadratic pseudo-potentials, [6], have been found. The conjecture of linearizability of equation (1) in the case $\kappa = -1$ has been made in [4]. In [7], a formula for the general solution of (1) has been proposed. This formula uses a nonlocal change of variables.

In this paper, we prove that equation (1) is equivalent under a contact transformation to the Euler–Poisson equation, [8, § 9.6],

$$u_{tx} = \frac{1}{\kappa\,(t+x)}\, u_t + \frac{2\,(1-\kappa)}{\kappa\,(t+x)}\, u_x - \frac{2\,(1-\kappa)}{(\kappa\,(t+x))^2}\, u, \tag{2}$$

and find the general solution of (1) in terms of local variables.

In [9], É. Cartan's method of equivalence, [10]–[12], [13, 14], in its form of the moving coframe method, [15, 16, 17], was used to find the Maurer–Cartan forms for the pseudo-group of contact symmetries of equation (2). The structure equations for the symmetry pseudo-group have the form

$$\begin{aligned}
\mathrm{d}\theta_0 &= \eta_1 \wedge \theta_0 + \xi^1 \wedge \theta_1 + \xi^2 \wedge \theta_2,\\
\mathrm{d}\theta_1 &= \eta_2 \wedge \theta_1 - 2\,(1-\kappa)\,\theta_0 \wedge \xi^2 + \xi^1 \wedge \sigma_{11},
\end{aligned}$$

$$
\begin{aligned}
\mathrm{d}\theta_2 &= (2\,\eta_1 - \eta_2) \wedge \theta_2 - \theta_0 \wedge \xi^1 + \xi^2 \wedge \sigma_{22}, \\
\mathrm{d}\xi^1 &= (\eta_1 - \eta_2) \wedge \xi^1, \\
\mathrm{d}\xi^2 &= (\eta_2 - \eta_1) \wedge \xi^2, \\
\mathrm{d}\sigma_{11} &= (2\,\eta_2 - \eta_1) \wedge \sigma_{11} + \eta_3 \wedge \xi^1 + 3\,(2\,\kappa - 1)\,\theta_1 \wedge \xi^2, \\
\mathrm{d}\sigma_{22} &= (3\,\eta_1 - 2\,\eta_2) \wedge \sigma_{22} + \eta_4 \wedge \xi^2, \\
\mathrm{d}\eta_1 &= (2\,\kappa - 1)\,\xi^1 \wedge \xi^2, \\
\mathrm{d}\eta_2 &= (1 - 4\,\kappa)\,\xi^1 \wedge \xi^2, \\
\mathrm{d}\eta_3 &= \pi_1 \wedge \xi^1 - (2\,\eta_1 - 3\,\eta_2) \wedge \eta_3 + 4\,(3\,\kappa - 1)\,\xi^2 \wedge \sigma_{11}, \\
\mathrm{d}\eta_4 &= \pi_2 \wedge \xi^2 + (4\,\eta_1 - 3\,\eta_2) \wedge \eta_4 + 2\,(3 - \kappa)\,\xi^1 \wedge \sigma_{22},
\end{aligned}
\tag{3}
$$

where θ_0, θ_1, θ_2, ξ^1, ξ^2, σ_{11}, σ_{22}, η_1, ... , η_4 are the Maurer–Cartan forms, while π_1 and π_2 are prolongation forms. We have $\theta_0 = a\,(\mathrm{d}u - u_t\,\mathrm{d}t - u_x\,\mathrm{d}x)$, $\theta_1 = a\,b^{-1}(\mathrm{d}u_t - u_{tt}\mathrm{d}t - R_2\,\mathrm{d}x) + 2\,(\kappa - 1)\,(\kappa\,b\,(t+x))^{-1}\theta_0$, $\theta_2 = a\,b\,\kappa\,(t+x)^2\,(\mathrm{d}u_x - R_2\,\mathrm{d}t - u_{xx}\,\mathrm{d}x) + b\,(t+x)\,\theta_0$, $\xi^1 = b\,\mathrm{d}t$, and $\xi^2 = b^{-1}\kappa^{-1}(t+x)^{-2}\mathrm{d}x$, where R_2 is the right-hand side of equation (2), while a and b are arbitrary non-zero constants. The forms σ_{11}, ... , π_2 are too long to be written out in full here. We write equation (1) and its Maurer–Cartan forms in tilded variables, then similar computations give $\widetilde{\theta}_0 = \widetilde{a}\,(\mathrm{d}\widetilde{u} - \widetilde{u}_{\widetilde{t}}\mathrm{d}\widetilde{t} - \widetilde{u}_{\widetilde{x}}\,\mathrm{d}\widetilde{x})$, $\widetilde{\theta}_1 = \widetilde{a}\,\widetilde{b}^{-1}(\mathrm{d}\widetilde{u}_{\widetilde{t}} - \widetilde{u}_{\widetilde{t}\widetilde{t}}\mathrm{d}\widetilde{t} - \widetilde{R}_1\,\mathrm{d}\widetilde{x}) - \widetilde{b}^{-2}\widetilde{u}\,\widetilde{u}_{\widetilde{x}\widetilde{x}}\,\widetilde{\theta}_2 - (2\,\kappa - 1)\,\widetilde{b}\,\widetilde{u}_{\widetilde{x}}\,\widetilde{\theta}_0$, $\widetilde{\theta}_2 = \widetilde{a}\,\widetilde{b}^{-1}(\widetilde{u}_{\widetilde{x}\widetilde{x}})^{-1}\,(\mathrm{d}\widetilde{u}_{\widetilde{x}} - \widetilde{R}_1\,\mathrm{d}\widetilde{t} - \widetilde{u}_{\widetilde{x}\widetilde{x}}\,\mathrm{d}\widetilde{x})$, $\widetilde{\xi}^1 = \widetilde{b}\,\mathrm{d}\widetilde{t}$, and $\widetilde{\xi}^2 = \widetilde{b}^{-1}\,(\mathrm{d}\widetilde{u}_{\widetilde{x}} - \kappa\,(\widetilde{u}_{\widetilde{x}})^2\,\mathrm{d}\widetilde{t})$, where $\widetilde{R}_1$ is the right-hand side of equation (1) written in the tilded vatiables, while $\widetilde{a}$ and $\widetilde{b}$ are arbitrary non-zero constants. The forms $\widetilde{\sigma}_{11}$, ... , $\widetilde{\pi}_2$ are too long to be written out in full. The structure equations for (1) differ from (3) only in replacing θ_0, ... , π_2 by their tilded counterparts. Therefore, results of Cartan's method (see, e.g., [14, th 15.12]) yield the contact equivalence of equations (1) and (2). Since the Maurer–Cartan forms for both symmetry groups are known, the equivalence transformation $\Psi : (t, x, u, u_t, u_x) \mapsto (\widetilde{t}, \widetilde{x}, \widetilde{u}, \widetilde{u}_{\widetilde{t}}, \widetilde{u}_{\widetilde{x}})$ can be found from the requirements $\Psi^*\widetilde{\theta}_0 = \theta_0$, $\Psi^*\widetilde{\theta}_1 = \theta_1$, $\Psi^*\widetilde{\theta}_2 = \theta_2$, $\Psi^*\widetilde{\xi}^1 = \xi^1$, and $\Psi^*\widetilde{\xi}^2 = \xi^2$:

Theorem. *The contact transformation* Ψ

$$
\begin{aligned}
\widetilde{u} &= (t+x)^{-\frac{1}{\kappa}}\left(\kappa\,(t+x)\,u_x + (\kappa - 1)\,u\right), \\
\widetilde{t} &= \kappa^{-1}\,t, \\
\widetilde{x} &= -(t+x)^{\frac{\kappa-1}{\kappa}}\left(\kappa\,(t+x)\,u_x - u\right), \\
\widetilde{u}_{\widetilde{t}} &= \kappa^2\,(t+x)^{-\frac{1}{\kappa}}\left(u_t - u_x\right), \\
\widetilde{u}_{\widetilde{x}} &= -(t+x)^{-1}
\end{aligned}
$$

takes the Euler–Poisson equation (2) to the generalized Hunter–Saxton equation (1) (written in the tilded variables).

Remark. The equivalence transformation Ψ is not uniquely determined: for any Φ and Υ from (isomorphic) infinite-dimensional pseudo-groups of contact symmetries of equations (1) and (2), respectively, the transformation $\Phi \circ \Psi \circ \Upsilon$ is also an equivalence transformation.

Equation (2) belongs to the class of linear hyperbolic equations $u_{tx} = T(t,x)\,u_t +$

$X(t,x)\,u_x + U(t,x)\,u$ and has important features: it has an intermediate integral, and its general solution can be found in quadratures. To prove this, we compute for equation (2) the Ovsiannikov invariants, [8, § 9.3], $P = K\,H^{-1}$ and $Q = (\ln|H|)_{tx}\,H^{-1}$, where $H = -T_t + T\,X + U$ and $K = -X_x + T\,X + U$ are the Laplace semi-invariants. We have $P = 2\,(1-\kappa)$ and $Q = 2\,\kappa$, therefore $P + Q = 2$, and the Laplace t-transformation, [8, § 9.3], takes equation (2) to a factorizable linear hyperbolic equation. Namely, we consider the system

$$v \;= u_x - (\kappa\,(t+x))^{-1}u, \tag{4}$$

$$v_t = 2\,(1-\kappa)\,(\kappa\,(t+x))^{-1}v + \kappa^{-1}\,(t+x)^{-2}u. \tag{5}$$

Substituting (4) into (5) yields equation (2), while expressing u from (5) and substituting it into (4) gives the equation

$$v_{tx} = \frac{1-2\,\kappa}{\kappa\,(t+x)}\,v_t + \frac{2\,(\kappa-1)}{\kappa\,(t+x)}\,v_x - \frac{(2\,\kappa-1)\,(\kappa-2)}{(\kappa\,(t+x))^2}v \tag{6}$$

with the trivial Laplace semi-invariant H. Hence, the substitution

$$w = v_x + (2\,\kappa - 1)\,(\kappa\,(t+x))^{-1}v \tag{7}$$

takes equation (6) into the equation

$$w_t = -2\,(\kappa-1)\,(\kappa\,(t+x))^{-1}w. \tag{8}$$

Integrating (8) and (7), we have the general solution for equation (6):

$$v = (t+x)^{\frac{1-2\kappa}{\kappa}}\,\left(S(t) + \int R(x)\,(t+x)^{\frac{1}{\kappa}}\,\mathrm{d}x\right),$$

where $S(t)$ and $R(x)$ are arbitrary smooth functions of their arguments. Then equation (5) gives the general solution for equation (2):

$$u{=}(t{+}x)^{\frac{1}{\kappa}}\,\left(\kappa\,S'(t){+}\int R(x)\,(t{+}x)^{\frac{1-\kappa}{\kappa}}\,\mathrm{d}x\right) - (t{+}x)^{\frac{1-\kappa}{\kappa}}\,\left(S(t){+}\int R(x)\,(t{+}x)^{\frac{1}{\kappa}}\,\mathrm{d}x\right).$$

This formula together with the contact transformation of the theorem gives the general solution for the generalized Hunter–Saxton equation (1) in a parametric form:

$$\begin{aligned}
\widetilde{u} &= \kappa^2\,S'(t) + \kappa\int R(x)\,(t+x)^{\frac{1-\kappa}{\kappa}}\,\mathrm{d}x,\\
\widetilde{t} &= \kappa^{-1}\,t,\\
\widetilde{x} &= -\kappa\,\left(S(t) + \int R(x)\,(t+x)^{\frac{1}{\kappa}}\,\mathrm{d}x\right).
\end{aligned}$$

Hence, we obtain the general solution of equation (1) without employing nonlocal transformations.

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